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PROCEEDINGS OF COLLOQUIUM ON STABLE SOLUTIONS OF SOME ILL-POSED--ETC(U)

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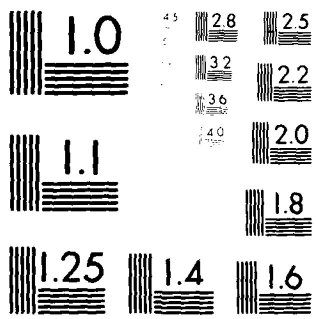

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This report contains the proceedings of a colloquium at the Naval Research Laboratory which was based upon Professor Ramm's lecture at the International Symposium on Ill-posed Problems. Stable solutions of equations of the first kind and equations of the second kind at a characteristic value are given. Iterative processes for solutions are constructed. Extension of operators is used to make of an ill-posed problem a well-posed one.

CONTENTS

PROCEEDINGS OF COLLOQUIUM ON STABLE SOLUTIONS OF SOME ILL-POSED PROBLEMS

§1. Introduction

Consider the following operator equation

$$Af = g, \tag{0}$$

where $A: X \to Y$ is a mapping from a Banach space X into a Banach space Y, $|f|$, $|g|$ are norms in X and Y respectively. If the inverse mapping A^{-1} exists and is continuous then the problem of finding $f \in X$ from equation (1) is called well-posed. In many cases $g \in R(A)$, $R(A)$ is the range of A, operator A^{-1} exists but is not continuous and is defined not on all Y. Then the problem (1) is called ill-posed, because f does not depend on g continuously. In what follows we assume that A is a linear bounded operator defined on X, $\mathrm{Ker}\, A = \{0\}$, $g \in R(A)$, A^{-1} is unbounded. We are not going to present here the most general results, but rather concentrate on some special situations which are important in applications. Another feature of this presentation consists in the following. We do not use general methods for stable calculation of the solution of equation (1) such as regularization techniques or the method of quasisolutions ([1]-[5]), because these methods are applicable in very general situations and that is why the characteristic features of some special classes of problems are not taken into account in the process of applying these methods. On the other hand, these general methods are not easy to apply from the standpoint of numerical analysis. The methods used in what follows are either iterative or based on some explicit analytical formulas for approximate solutions. Yet another point to be emphasized consists in the following. Suppose that the mapping

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A is compact, $X = Y = H$ where H is a Hilbert space. Operator A^{-1} is unbounded. Thus the problem (1) is ill-posed. But there are some situations when it is natural to seek the solution not in H but in a wider space H_-. Operator A in these situations can be extended to the operator A: $H_- \to H_+$, $H_+ \subset H \subset H_-$, the embeddings are dense in the next space, and the extension of A is a homomorphism from H_- onto H_+. Hence we made of an ill-posed problem a well-posed one. An example of such a construction is given in §6. The space H_- in §8 is the space of distributions. Basic problems of the static fields theory can be reduced to integral equations of the second kind at a characteristic value. This is another kind of ill-posed problem similar to ill-conditioned problems in linear algebra but not identical to them.

In §2 we give a simple remark of general nature. In §3 an iterative process for solution of integral equations with positive kernels is given. In §4 an iterative process for solution of integral equation of interest in geophysics is given. In §5 an iterative process for stable solution of the operator equation $f = \lambda_1 Af + g$ will be given, λ_1 is a characteristic value of A. In §6 a study of integral equations basic for estimation theory, filtering and prediction of random fields will be given. It is in this case that one should look for a solution in the space of distributions. In §7 some ill-posed problems are discussed which can be naturally treated as the problems of approximation theory. In §8 optimal solutions of some ill-posed problems will be discussed. In §9 equation with many solutions is treated.

In the bibliographical notes we discuss some other trends in the theory of ill-posed problems and give references. We use

autonomous numeration in each section, but references to formulas of other section have the number of section.

This introduction we end with examples.

Example 1. Let F be a perfectly conducting surface in $\mathbb{R}^3$ not necessarily closed. Then equation

$$\int_F \sigma(t)\ (4\pi\varepsilon_o r_{st})^{-1}\, dt = 1,\ r_{st} = |s-t| \tag{1}$$

describes the equilibrium charge distribution σ on this surface provided that the potential on the surface is equal to 1, $\varepsilon_o > 0$ is a constant. Equation (1) is an integral equation of the first kind.

Example 2. If a perfect conductor D with a surface Γ is placed in the electrostatic field $-\nabla u$ then the charge distribution σ on Γ satisfies integral equation

$$\sigma = -A\sigma - 2\varepsilon_o\ \partial u/\partial N,\ A\sigma = \int_\Gamma \sigma(t)\ \frac{\partial}{\partial N_s}\frac{1}{2\pi r_{st}}\, dt, \tag{2}$$

N is the outer normal to Γ.

It is known that -1 is a characteristic value of the operator A. Also it is known, that equation (2) is solvable, if $\int_\Gamma \partial u/\partial N\, dt = 0$. But this condition is satisfied because $\Delta u = 0$. Equation (2) is so called an equation on spectrum.

Similar equations are valid in other static fields theories, e.g. in hydrodynamics, magnetostatics, elasticity.

Example 3. Equation

$$\int_{-1}^{1} (x-t)^{-1} f(t)dt = g(x),\ x > 1 \tag{3}$$

is of interest in geophysics. Its multidimensional analogue is

$$\int_D f(t)\ r_{xt}^{-1}\, dt = g(x),\ x\varepsilon\Delta,\ \Delta\cap D = \emptyset \tag{4}$$

Example 4. Basic problems of random fields filtering and estimation theory can be reduced to equation

$$\int_D R(x,y)h(y)\,dy = f(x), \quad x\in\bar{D} = D + \partial D = D + \Gamma, \tag{5}$$

where $R(x,y)$ is a correlation function. In this theory the solution of equation (5) under natural assumptions concerning the kernel $R(x,y)$ is to be found in a class of distributions.

Example 5. Some problems of antenna synthesis and apodization theory can be reduced to equation

$$\int_D \exp\{i(x,y)\}\, j(y)\,dy = f(x), \quad x\in\Delta. \tag{6}$$

In optics apodization means transformation of the electromagnetic field on the output pupil of optical instrument in order to make the image better, to improve resolution of the instrument.

Example 6. In monoimpulse radiolocation the direction-finding characteristic (pelengation characteristic) is defined by the formula

$$g(k) = f^2\,(k+k_o) - f^2(k-k_o), \tag{7}$$

where

$$f(k) = \int_{-\pi}^{\pi} j(x)\, \exp(ikx)\,dx \tag{8}$$

is the radiation pattern of a linear antenna, $j(x)$ is the current distribution and $0 < k_o < \pi/2$ is a small number. Equation (7) is a nonlinear integral equation for $j(x)$.

The stable approximation to the solution of equation (0) can be described as follows. Assume that g is known with the error $\delta > 0$, i.e. an element g_δ: $|g_\delta - g| \leq \delta$ is known. The problem is to calculate f_δ so that $|f_\delta - f| \to 0$ as $\delta \to 0$ and in some sense f_δ is a solution of the equation (0). For example one can require that $|Af_\delta - g| \to 0$ as $\delta \to 0$.

§2. Iterative methods of stable solutions of ill-posed problems.

In what follows we omit the sign of space in the notations of the norms since it is clear what norm we use. Suppose that an iterative process is known for solution of equation (1), $f_n = B_n g \to f = A^{-1}g$ as $n \to \infty$, and the operator B_n is continuous for any fixed n. Then it is possible to give a stable approximation to the solution of equation (1).

Indeed, if an element g_δ, $|g - g_\delta| \le \delta$ is given we take $f_{n(\delta)} = B_{n(\delta)} g_\delta$ and show that it is possible to choose $n(\delta)$ so that $|f_{n(\delta)} - f| \to 0$ as $\delta \to 0$. To this end we start with the inequality:

$$|B_n g_\delta - f| \le |B_n g - B_n g| + |B_n g - f|. \tag{1}$$

According to our assumptions

$$a(n) \equiv |B_n g - f| \to 0 \text{ as } n \to \infty, \tag{2}$$

$$|B_n g_\delta - B_n g| \le b(\delta,n) \to 0 \text{ as } \delta \to 0 \tag{3}$$

for any fixed n. Thus

$$|B_n g_\delta - f| \le a(n) + b(\delta,n). \tag{4}$$

Let us take for any $\delta > 0$ such $n = n(\delta)$ that

$$a(n) + b(\delta,n) = \min \equiv \alpha(\delta) \tag{5}$$

From (2), (3) it follows that

$$n(\delta) \to \infty \text{ as } \delta \to 0,\ \alpha(\delta) \to 0 \text{ as } \delta \to 0. \tag{6}$$

It follows from (6) and (4) that

$$|B_{n(\delta)} g_\delta - f| \to 0 \text{ as } \delta \to 0. \tag{7}$$

We proved a simple but useful theorem.

Theorem 1. If an iterative process for solution of equation (1) is known, $f_n = B_n g$ is the n-th approximation given by the iterative process and B_n is a continuous operator, then it is possible to give a stable approximation of the solution of equation (1).

Remark 1. If $B_n = \sum_{j=0}^{n} T^j$, where T is a linear bounded operator, then $|B_n g_\delta - B_n g| \le \frac{1 - ||T||^{n+1}}{1 - ||T||} \delta \equiv b(\delta,n)$. Thus in this case there is an explicit formula for $b(\delta,n)$.

Remark 2. In proposition 1 no use was made of the linearity of A.

§3. Iterative process for the solution of integral equations of the first kind with positive kernels.

Here we give an iterative process for solution of integral equations of the first kind with positive kernels. Equation (1.1) belongs to this class but also some nonself-adjoint equations are in this class.

Consider the equation

$$Kf = \int_D K(x,y)f(y)dy = g(x), \quad x \in D \tag{1}$$

where $D \subset R^r$ is a bounded domain, the operator $K: \ H \to H$, $H = L^2(D)$, is compact, $K(x,y) > 0$. Assume that a function $h(x) > 0$ exists such that

$$Kh \le c, \ \int_D a(x)dx < \infty, \ a(x) \ge m > 0, \tag{2}$$

where

$$a(x) \equiv h(x) \ (Kh)^{-1} \tag{3}$$

Denote by $\phi = f(x)a^{-1}(x)$ and by $H_\pm$ the spaces $L^2(D, a^{\pm 1}(x))$ with the norms $|g|_\pm^2 = \int_D |g|^2 \ a^{\pm 1}(x)dx$. Equation (1) can be written as

$$K_1\phi = g, \ K_1\phi \equiv \int_D K(x,y)a(y)\phi(y)dy. \tag{4}$$

Let K_1 be a compact operator in H_+,

$$K_1\phi_j = \lambda_j\phi_j, \ |\lambda_1| > |\lambda_2| \ge \ldots \tag{5}$$

The kernel in (4) is positive. We suppose that some iteration of K_1 is continuous. It is known that such kernels have a unique positive eigenvalue λ_1 with the positive corresponding eigenfunction and moduli of all other eigenvalues are strictly less than λ_1 (see [12], theorem 4.1.4). We rewrite equation (4) as

$$\phi = Q\phi + g, \quad Q \equiv I - K_1, \tag{6}$$

and note that

$$K_1 Kh = Kh. \tag{7}$$

Thus operator K_1 has a positive eigenfunction Kh which corresponds to a positive eigenvalue $\lambda_1 = 1$. Hence all other eigenvalues of K_1 satisfy the inequality

$$|\lambda_j| < 1, \quad j = 2, 3, \ldots \tag{8}$$

Consider the iterative process

$$\phi_{n+1} = Q\phi_n + g, \quad \phi_0 = g. \tag{9}$$

Theorem 1. Suppose that the above assumptions about $K_1(x,y)$ hold, and i) equation (4) is solvable in H_+, ii) the system $\{\phi_j\}$ forms a Riesz basis in H_+ and $\lambda_j \neq 0$, $|\arg\lambda_j| \leq \pi/3$, $j = 2, 3, \ldots$.

Then process (9) converges in H_+ to a solution ϕ of equation (4), equation (1) is solvable in H_- and its solution is $f = \phi(x)a^{-1}(x)$.

Remark 1. We remind that a system $\{\phi_j\}$ forms a Riesz basis of a Hilbert space H if the system is complete and minimal in H and for any numbers $C_1, \ldots, C_n$ the inequality

$$a \sum_1^n |c_j|^2 \leq |\sum_1^n c_j\phi_j|^2 \leq b\sum_1^n |c_j|^2 \tag{10}$$

holds, where $0 < a < b$ are constants which do not depend on n (see [13]).

Remark 2. According to Proposition 1 iterative process (9) can be used for the stable approximation of the solution of equation (1).

Remark 3. In applications the freedom of choice of $a(x)$ can be useful. For example the solution of equation (1.1) for an unclosed surface (thin perfectly conducting screen) must have the known singularity near the edge L of F: $\sigma \sim \rho^{-1/2}(x)$, where $\rho(x) = \min_{s\in L}|x-s|$. If we take $a(t) = \rho^{-1/2}(t)$ then the function $\phi = \rho^{1/2}(t)\sigma(t)$ will be continuous and the approximate solution of equation (1.1), obtained by means of iterative process given in Theorem 1, will have the right singularity for any n. It is clear that the kernel of equation (1.1) satisfies conditions of Theorem 1.

Remark 4. In [24] iterative process (9) was applied for calculation of the magnetization of thin magnetic films. This problem is of interest for computer technology.

Proof of Theorem 1. Let $\delta_n = \phi - \phi_n$. Then $\delta_n = Q^n\delta_0$. Let $\delta_0 = \sum_1^\infty C_j\Phi_j$. This expansion is possible because the system $\{\Phi_j\}$ forms a Riesz basis of H_+. We have $\delta_n = \sum_1^\infty (1-\lambda_j)^n C_j\Phi_j$,

$|\delta_n|_{H_+}^2 \le b \sum_1^\infty |1-\lambda_j|^{2n}|C_j|^2 \to 0$ as $n \to \infty$.

Here we took into account that $|\arg\lambda_j| \le \pi/3$ implies $|1-\lambda_j| < 1$. Indeed, if $\lambda = r\exp(i\psi)$, $r < 1$, $|\psi| \le \pi/3$, then $|1-\lambda|^2 = 1+r^2 - 2r\cos\psi \le 1+r^2 - r < 1$. We proved that $\lim\phi_n = \phi$ exists in H_+. It is clear that ϕ is the solution of (4) and $f = a\phi\in H_-$ and is the solution of (1).

§4 Iterative process for the solution of an itegral equation of interest in geophysics.

Consider the equation

$$\int_{-1}^{1} (x-t)^{-1} f(t) = g(x), \; x > 1. \tag{1}$$

Its multidimensional analogue

$$\int_D |x-t|^{-1} f(t)\,dt = g(x), \; x \in \Delta, \; \Delta \cap D = \emptyset \tag{2}$$

can be intepreted as the problem of finding mass distribution from the known potential.

We will discuss equation (1) in detail and give some comments concerning equation (2). Note that the domain of integration and the domain where $g(x)$ is known are different. Let us set

$$x = N+y, \; -1 \le y \le 1, \; g(N+y) \equiv h(y), \tag{3}$$

where $N > 3$ will be chosen later. Equation (1) can be written as

$$Af \equiv \int_{-1}^{1} (N+y-t)^{-1} f(t)\,dt = h(y), \; -1 \le y \le 1. \tag{4}$$

This equation is equivalent to equation $A^*Af = A^*h$, which we write as:

$$Kf = \psi, \; \psi = A^*h, \; K = A^*A, \tag{5}$$

where the kernel of K is

$$K(z,t) = \int_{-1}^{1} \frac{dy}{(N+y-t)(N+y-z)} = \frac{1}{t-z} \ln \frac{(N+1-t)(N-1-z)}{(N+1-z)(N-1-t)} \tag{6}$$

We use here the known lemma (see, e.g., [7]).

Lemma 1. Let $A: H \to H$ be a linear operator on a Hilbert space, $g \in D(A^*) \cap R(A)$. Then equation $Af=g$ is equivalent to equation $A^*Af = A^*g$.

Proof of Lemma 1. If $Af = g$ and $g \in D(A^*)$ then $A^*Af = A^*g$. If $A^*Af = A^*g$ and $g \in R(A)$ then $A^*Af = A^*Ah$. Thus $(A^*A(f-h), f-h) = 0$, $Af = Ah = g$.

Operator K is self-adjoint and nonnegative in $H = L^2[-1, 1]$.
Equation (5) is equivalent to equation

$$f = (I - K)f + \psi. \tag{7}$$

From (6) it follows that

$$K(z,t) \sim 2N^{-2} \text{ as } N \to \infty. \tag{8}$$

Hence the operator $B = I - K$ is positive definite for large N and $\|B\| \le 1$. Since $K = K^*$ is compact we conclude that $\|B\| = 1$. Indeed

$$\|B\| = \sup_{|f| \le 1} |f - Kf| \ge |\phi_n - K\phi_n| = 1 - \lambda_n \to 1 \text{ as } n \to \infty. \tag{9}$$

Here ϕ_n, λ_n are the eigenfunctions and eigenvalues of K. In [7] the following theorem is proved.

Theorem (M. Krasnoselskij). Let $B = B^*$, $\|B\| = 1$ be a linear operator on a Hilbert space H and -1 is not an eigenvalue of B. Then the iterative process

$$f_{n+1} = Bf_n + g, \quad f_0 \varepsilon H \tag{10}$$

converges in H to a solution of the equation $f = Bf + g$ if this equation has a solution.

In our problem operator $B = I - K$ satisfies the conditions of the theorem. Thus we have proved Theorem 1. If equation (1) has a solution in H then iterative process

$$f_{n+1} = (I-K)f_n + \psi, \quad f_0 \varepsilon H \tag{11}$$

converges in H and $f = \lim f_n$ is the solution of equation (1) provided that N is large enough (so that $\|K\| \le 1$).

Remark 1. Since g is analytic outside the cut [-1, 1] it is uniquely determined everywhere outside this cut. The function f is

uniquely determined from the jump relations for the Cauchy integral

Remark 2. According to Theorem 1 from §2 iterative process (11) gives a possibility to construct a stable approximation to the solution of equation (1).

Remark 3. The method of solution of equation (1) holds for equation (2) without essential alternations, but the question about uniqueness of solution of equation (2) is more difficult. Under some additional assumptions about the solution of equation (2) uniqueness of the solution can be proved [14].

From the Krasnoselskij theorem the following proposition follows. Let $Af = g$, A is a bounded linear operator on H, $g \varepsilon R(A)$. Then the iterative process

$$f_{n+1} = (I - \alpha A^*A)f_n + \alpha A^*g, \quad f_0 \varepsilon H, \quad 0 < \alpha < \frac{2}{\|A^*A\|} \tag{12}$$

converges in H to a solution of equation $Af = g$ for any $f_0 \varepsilon H$.

§5. <u>Iterative process for the solution of the equation of the second kind at a characteristic value.</u>

If a perfect conductor is placed in the initial field with the potential $-\nabla u$ then the charge distribution on the surface of the conductor satisfies equation (1.2). The number -1 is the smallest characteristic value of the operator A in equation (1.2) and this equation is solvable since $\int_\Gamma \partial u/\partial N \, dt = 0$. Since equation (1.2) is an equation at a characteristic value (equation on spectrum we shall write in the sequel), the problem (1.2) is ill-posed. Similar equations occur in hydrodynamics of ideal incompressible fluid and in elasticity. In all these cases the

characteristic value is semi-simple. It means that the root space corresponding to this characteristic value coincides with its eigenspace. We present now the abstract setting. Let A be a linear operator on a Hilbert space H. Suppose that the spectrum of A is discrete, λ_j are the characteristic values of A, $\phi_j = \lambda_j A\phi_j$, $|\lambda_1| < |\lambda_2| \leq |\lambda_3| \leq \ldots$. Suppose that λ_1 is semi-simple. Denote by $G = \mathrm{Ker}\,(I - \bar{\lambda}_1 A^*)$, by $\{\psi_j\}_{j=1}^{m}$ an orthonormal basis of G, by $G^{\perp} = R(I - \lambda_1 A)$, where R(A) is the range of A, by P orthoprojector in H onto G. We consider the equation

$$g = \lambda_1 Ag + f, \; f \varepsilon G^{\perp}, \tag{1}$$

and introduce the following operator

$$B_{\gamma} g = Ag + \gamma \sum_{j=1}^{m} (g, \psi_j)\psi_j. \tag{2}$$

Denote by r_{γ} the following number

$$r_{\gamma} = \min\,(|\lambda_2|, |\lambda_1(1 + \gamma\lambda_1)^{-1}|), \tag{3}$$

where γ is a number which we fix later.

Equation

$$g = \lambda_1 B_{\gamma} g + f \tag{4}$$

is equivalent to equation (1) on the set $G^{\perp}$. It means that every solution $g \varepsilon G^{\perp}$ of (1) is a solution of (4) and vice versa. Let us assume that $\dim G = \dim \ker\,(I - \lambda_1 A)$. This is so if e.g. A is compact.

Theorem 1. Operator B_{γ} has no characteristic values in the disk $|\lambda| < r_{\gamma}$. If $|1 + \gamma\lambda_1| < 1$ then the iterative process

$$g_{n+1} = \lambda_1 B_{\gamma} g_n + F, \; g_0 = F, \tag{5}$$

converges not slower than geometrical progression with denominator q, $0 < q < |\lambda_1|\, r_{\gamma}^{-1}$ to the solution of the equation $g = \lambda_1 B\, g + F$, $\forall F \varepsilon H$.

If $F \varepsilon G^{\perp}$ then F can be represented in the form $F = \lambda_1 Av - v$ where v is some element of H. In this case $g = \Phi - v$, with $\Phi \varepsilon$ Ker $(I - \lambda_1 A)$ and $P\Phi = Pv$. If dim $G = 1$, $\phi \varepsilon$ Ker $(I - \lambda_1 A)$, $\psi \varepsilon G$, $\|\phi\| = \|\psi\| = 1$, then $\Phi = \phi(f,\psi)/(\phi,\psi)$. Process (5) is stable in the following sense: the sequence

$$h_{h+1} = \lambda_1 B_\gamma h_n + F + \varepsilon_n, \; h_0 := F, \; \|\varepsilon_n\| \leq \varepsilon \tag{6}$$

satisfies the estimate

$$\lim\sup \|g - h_n\| = O(\varepsilon) \text{ as } n \to \infty, \tag{7}$$

where $g = \lim g_n$ as $n \to \infty$.

Theorem 2. The iterative process

$$f_{n+1} = \lambda_1 A f_n, \; f_0 = f \tag{8}$$

converges at the rate of geometrical progression with denominator q, $0 < q < |\lambda_1 \lambda_2^{-1}|$ to an element of Ker $(I - \lambda_1 A)$. If dim $G = 1$ then this element is equal to $\phi(f,\psi)/(\phi,\psi)$.

Remark 1. The problems of static fields theory (electrostatic, hydrodynamics, elasticity) lead to equation (1). In electrostatics (equation (1.2)) dim $G = 1$, in elasticity and hydrodynamics dim $G =$ $= 6$. In electrostatics when the boundary Γ has n connected components (the case of n bodies in the exterior field) dim $G = n$.

Proof of Theorem 1. Let $g = \lambda B_\gamma g$. Multiplying this equation by ψ_j we get $(g, \psi_j) = \lambda\lambda_1^{-1}(g, \psi_j) + \lambda\gamma(g, \psi_j)$. Thus $(g, \psi_j)(1 - \lambda\lambda_1^{-1} - \lambda\gamma) = 0$. If $(g, \psi_j) \neq 0$ for some j, $1 \leq j \leq m$, then $\lambda = \lambda_1 (1+\gamma\lambda_1)^{-1}$. If $(g, \psi_j) = 0$ for $1 \leq j \leq m$, then $B_\gamma g = Ag$ and, therefore, $\lambda \varepsilon \sigma(A)$ ($\sigma(A)$ is the spectrum of A). From this we conclude that in the disk $|\lambda| < r_\gamma$, where r_γ is defined by formula (3), there are no characteristic values of B_γ. Indeed,

if $\lambda_1 \varepsilon \sigma(B_\gamma)$ then $(g, \psi_j) = 0$ for $1 \leq j \leq m$, $g \neq 0$ and hence $g = \lambda_1 Ag$, $g \varepsilon G^\perp$. This is impossible because λ_1 is semisimple. (We remind the reader that λ_1 is semisimple iff $(I - \lambda_1 A)^2 g = 0 \Rightarrow$ $\Rightarrow (I - \lambda_1 A) g = 0$, or iff $\{(I - \lambda_1 A) g = 0, g \varepsilon G^\perp\} \Rightarrow g = 0$, or iff λ_1 is a simple pole of $(I - \lambda A)^{-1}$). Hence $\lambda_1 \notin \sigma(B_\gamma)$. Spectrum of B_γ consists of $\sigma(A) \setminus \lambda_1$ and possibly of number $\lambda_1(1 + \gamma\lambda_1)^{-1}$. If $\Phi = \lambda B_\gamma \Phi$, $\lambda \neq \lambda_1$, then $\Phi \varepsilon G^\perp$, $\Phi = \lambda A \Phi$ and therefore $\lambda \varepsilon \sigma(A)$.

Let us prove that process (5) converges. If we choose γ so that $|1 + \gamma\lambda_1| < 1$, then $q = |\lambda_1| r_\gamma^{-1} < 1$ and we conclude that there are no characteristic values of B_γ in the disk $|\lambda| < r$. Thus process (5) converges not slower than geometrical progression with denominator $q = |\lambda_1| r^{-1}$ $0 < q < 1$. If $F = \lambda_1 Av - v$ it is clear that $F \varepsilon G^\perp$. Thus $Af = B_\gamma F$. Let $g = \sum_{j=0}^{\infty} \lambda_1^j B_\gamma^j F$, then $g \varepsilon G^\perp$, $Ag = B_\gamma g$, $g = \lambda_1 B_\gamma g + F$. From this it follows that $g + v = \lambda_1 A(g + v)$, so that $g + v = \Phi \varepsilon$ Ker $(I - \lambda_1 A)$. Since $Pg = 0$ we conclude that $Pv = P\Phi$. If dim G = dim Ker $(I - \lambda_1 A) = 1$, $\phi \varepsilon G$, $\|\phi\| = \|\psi\| = 1$, then $\Phi = c\phi$, c = const. Multiplying this equality by ψ we obtain $c = (\phi, \psi)/(f, \psi)$, because $(g, \psi) = 0$ and $(\phi, \psi) \neq 0$. It remains to prove the stability of process (6), i.e. estimate (7). We have

$$h_n = \sum_{j=0}^{n} (\lambda_1 B_\gamma)^j F + \sum_{j=0}^{n-1} (\lambda_j B_\gamma)^j \varepsilon_{n-1-j},$$

$$g = \sum_{j=0}^{\infty} (\lambda_1 B_\gamma)^j F, \quad \|\lambda_1 B_\gamma\| \leq q.$$

From this it follows that

$$\|g - h_n\| \leq \sum_{j=0}^{n-1} q^j \varepsilon + \sum_{j=n+1}^{\infty} q^j \|F\| \leq (\varepsilon + \|F\| q^{n+1})(1-q)^{-1}. \tag{9}$$

Formula (8) follows from (9).

Proof of Theorem 2. First we prove the following lemma.

Lemma 1. Let a function $F(z)$ be analytic in the disk $|z| < r$ and meromorphic in the disk $|z| < r + \varepsilon$, $\varepsilon > 0$ with values in the algebra of linear bounded operators on a Hilbert space. Suppose that λ is a simple pole of $F(z)$, $\operatorname{res}_{z=\lambda} F(z) = c$, $|\lambda| = r$. If there are no other poles of $F(z)$ in the disk $|\lambda| < r + \varepsilon$ and

$F(z) = \sum_{h=0}^{\infty} a_n z^n$ for $|z| < r$, then $-\lim_{n\to\infty} \lambda^{n+1} a_n = c$.

Proof of Lemma 1. The function $F(z) - c(z - \lambda)^{-1}$ is analytic in the disk $|z| < r + \varepsilon$. Hence $F(z) - c(z - \lambda)^{-1} = \sum_{n=0}^{\infty} b_n z^n$, $|z| < r + \varepsilon$. For $|z| < r$ the identity holds: $\sum_{n=0}^{\infty} b_n z^n =$ $= \sum_{n=0}^{\infty} (a_n + c\lambda^{-n-1})z^n$. This identity is valid in the disk $|z| < r + \varepsilon$ according to analytic continuation principle. Thus $\lim_{n\to\infty} (\lambda^{n+1} a_n + c) = 0$.

Passing over to the proof of Theorem 2 we note, that the function $(I - zA)^{-1} f = \sum_{j=0}^{\infty} z^j A^j f$ is analytic in the disk $|z| < \lambda_1$, has a simple pole at $z = \lambda_1$ and has no other poles in the disk $|z| < \lambda_2$. Applying Lemma 1 we conclude that $\lim_{n\to\infty} (\lambda_1^{n+1} A^{n+1} f + c) = 0$. The rate of convergence is $0(|\lambda_1 \lambda_2^{-1}|^n)$. Since $f_n = \lambda_1^n A^n f$ we have $\lim_{n\to\infty} f_n = g$ exists. It is clear that $g = \lambda_1 Ag$. If $\dim \operatorname{Ker} (I - \lambda_1 A) = 1$ we have $g = a\phi$, $a = \text{const}$ and

$\phi \varepsilon$ Ker $(I - \lambda_1 A)$, $\phi \neq 0$. We note that $(f_{n+1}, \psi) = (\lambda_1 Af_n, \psi) =$ $= (f_n, \psi) = \ldots = (f, \psi)$. Thus $a(\phi,\psi) = (f,\psi)$, $a = (f,\psi)/(\phi,\psi)$.

Remark 2. Theorem 2 gives an abstract analogue of the iterative process for finding the equilibrium charge distribution on the surface of a perfect conductor.

§6. How to make of an ill-posed problem a well posed one.

In general there are several answers. One way is to substitute the original problem $Af = g$, $A: H \to H$ by some other problems (methods of quasi solutions and regularization). The other way is to consider an extension of the operator A, $A: H_- \to H_+$, $H_+ \subset H \subset H_-$ and to choose H_- and H_+ so that A will be a homomorphism of H_- onto H_+. In this case equation $Af = g$ has a unique and stable solution in H_-, $f = A^{-1}g$. We give an example of a class of integral equations which can be treated in such a way. The basic integral equation in random fields filtering and estimation theory can be written in the form

$$\int_D R(x,y)\, h(y)dy = f(x), \quad x \varepsilon \bar{D} \subset R^r, \tag{1}$$

where D is a bounded domain with a smooth closed boundary Γ, $\bar{D} = D + \Gamma$, $r > 1$.

Following [16] let us assume that the kernel $R(x,y) \varepsilon \mathcal{R}$. It means that a selfadjoint elliptic operator $L: L^2(R^r) \to L^2(R^r)$ and polynomials $P(\lambda) > 0$, $Q(\lambda) > 0$, $\lambda \varepsilon I = (-\infty,\infty)$ exist such that

$$R(x,y) = \int_\Lambda P(\lambda) Q^{-1}(\lambda) \Phi(x,y,\lambda) d\rho(\lambda) \tag{2}$$

Here Λ, Φ, $d\rho$ are the spectrum, spectral kernel and spectral measure of L respectively. Let p, q, s be deg P, deg Q and ord L

respectively, and $\alpha = 0.5(q-p)s$. Denote by H_t the scale of Sobolev spaces $H_t = W_2^t(D)$, H_{-t} for $t > 0$ is the dual space to H_t with respect to $H_0 = L^2(D)$. It was proved in [16]-[19] thatthe operator $R: H_{-\alpha} \to H_{\alpha}$ is a homomorphism of $H_{-\alpha}$ onto H_{α}. It means that the solution of equation (1) in $H_{-\alpha}$ exists, is unique and stable towards small perturbations of f in H_{α}.

Some explicit analytical formulas for the solution of equation (1) with kernel $R(x,y) \varepsilon R$ were obtained in [17]. This leads to the following method of stable solution of some integral equations. Given a kernel

$$R(x,y) = \int_{\Lambda} R(\lambda)\Phi(x,y,\lambda)d\rho(\lambda), \tag{3}$$

where $R(\lambda)$ is a function which satisfies the following condition:

$$0 < R(\lambda) \sim A(1 + \lambda^2)^{-\beta}, \quad \beta > 0 \text{ is integer.} \tag{4}$$

Then for any $\varepsilon > 0$ there exist polynomials $P_\varepsilon(\lambda)$, $Q_\varepsilon(\lambda)$, $\deg Q_\varepsilon(\lambda) - \deg P_\varepsilon(\lambda) = 2\beta$ such that

$$\sup_{\lambda \varepsilon I} \{(1 + \lambda^2)^{\beta} \, |R(\lambda) - R_\varepsilon(\lambda)|\} \equiv \|R(\lambda) - R_\varepsilon(\lambda)\| \le \varepsilon, \tag{5}$$

where

$$R_\varepsilon(\lambda) = P_\varepsilon(\lambda) \, Q_\varepsilon^{-1}(\lambda), \tag{6}$$

and $I = (-\infty,\infty)$. Consider the equations

$$Rh = f, \quad R_\varepsilon h_\varepsilon = f, \tag{7}$$

where the kernels of the operators R and R_ε are defined by the functions $R(\lambda)$ and $R_\varepsilon(\lambda)$ respectively.

Theorem 1. If (4) holds then for any $f \varepsilon H_{\beta s}$ there exist unique solutions $h, h_\varepsilon \,\varepsilon\, H_{-\beta s}$ of equations (7) and

$$|h - h_\varepsilon|_{-\beta s} \le 2M^2\varepsilon(1-2\varepsilon M)^{-1}|f|_{\beta s}, \tag{8}$$

provided that $2\varepsilon M < 1$, where

$$M = \{\inf_{\varepsilon>0} \inf_{\lambda\in I} [(1+\lambda^2)^\beta |R_\varepsilon(\lambda)|]\}^{-1}. \tag{9}$$

Proof of Theorem 1. We start with the following lemma.

Lemma 1. Let $H_+ \subset H_0 \subset H_-$ be a triple of Hilbert spaces, H_- is the dual space for H_+ with respect to H_0. Suppose that R: $H_- \to H_+$ is a linear map and

$$c_1 |h|_-^2 \le (Rh, h) \le c|h|_-^2, \; c_1 > 0, \forall h\varepsilon H_-. \tag{10}$$

Then R is a homomorphism of H_- onto H_+,

$$|R| \le 2c, \; |R^{-1}| \le c_1^{-1}, \tag{11}$$

where $|R|$ is the norm R: $H_- \to H_+$, $|h|_-$ is the norm in H_-.

Proof of Lemma 1. Operator R is defined on all H_- and from (10) it follows that

$$|Rh|_+ \ge c_1 |h|_- \tag{12}$$

Therefore R^{-1} is defined on $\mathrm{im}R \equiv$ range R and

$$|R^{-1}| \le c_1^{-1} \tag{13}$$

It remains to prove the first inequality (11). We have

$$|R| = \sup_{|g|_-\le 1, |h|_-\le 1} |(Rg, h)| = \sup_{|g|_-\le 1, |h|_-\le 1} \Big|\frac{1}{4} \{(R(h+g), h+g)$$

$$- (R(h-g),h-g) - i[(R(h+ig),h+ig) - (R(h-ig),h-ig)]\}\Big| \le$$

$$\le \sup_{|g|_-\le 1, |h|_-\le 1} \frac{c}{4} \{|h+g|_-^2 + |h-g|_-^2 + |h+ig|_-^2 + |h-ig|_-^2\} \le$$

$$\le c \sup_{|g|_-\le 1, |h|_-\le 1} \{|h|_-^2 + |g|_-^2\} \le 2c \tag{14}$$

It remains to prove that $\mathrm{im}R = H_+$. This is true because R is monotone, coercive and continuous (see [20]).

Lemma 2. Suppose that R_ε: $H_- \to H_+$ is a linear bijective mapping of H_- onto H_+ for any $\varepsilon > 0$, $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is some fixed number. Suppose that R: $H_- \to H_+$ is a linear mapping, $|R_\varepsilon^{-1}| \le M$, $|R - R_\varepsilon| < \varepsilon$, where $M > 0$ does not depend on ε, $0 < \varepsilon < \varepsilon_0$. Then R is a bijection of H_- onto H_+, and

$$|R^{-1}| \le M\ (1 - \varepsilon M)^{-1} \text{ for } \varepsilon M < 1 \tag{15}$$

Proof of Lemma 2. We have $R = R_\varepsilon[I + R_\varepsilon^{-1}\ (R - R_\varepsilon)]$, where I is the identity in H_-. The operator $R_\varepsilon^{-1}\ (R - R_\varepsilon)$ acts in H_- and $|R_\varepsilon^{-1}\ (R-R_\varepsilon)| \le \varepsilon M$. If $\varepsilon M < 1$ then $R^{-1} = [I + R_\varepsilon^{-1}\ (R - R_\varepsilon)]^{-1} R_\varepsilon^{-1}$ and

$$|R^{-1}| \le M\ (1 - \varepsilon M)^{-1} \tag{16}$$

Now it is possible to prove theorem 1. Using Parseval equality we conclude that

$$(Rh,h) = \int_\Lambda R(\lambda)|\tilde{h}(\lambda)|^2 dq(\lambda) \le c \int_\Lambda (1+\lambda^2)^{-\beta}|\tilde{h}|^2 dq(\lambda) = c|h|^2_{-\beta s}. \tag{17}$$

Here $\tilde{h}$ is the image of h in the representation generated by the expansion in eigenfunctions of the operator L, and

$$c = \sup_{\lambda \varepsilon I} \{(1+\lambda^2)^\beta\ R(\lambda)|\}. \tag{18}$$

If we denote

$$c_1 = \inf_{\lambda \varepsilon I} \{(1+\lambda^2)^\beta\ R(\lambda)\}, \tag{19}$$

then

$$c_1\ |h|^2_{-\beta s} \le (Rh,\ h) \le c|h|^2_{-\beta s}. \tag{20}$$

From (20) and Lemma 1 it follows that R is a bijection of $H_{-\beta s}$ onto $H_{\beta s}$ and (11) hold.

From (9) and Lemma 1 it follows that

$$|R_\varepsilon^{-1}| \le M. \tag{21}$$

From (5), (18) and (11) it follows that

$$|R - R_\varepsilon| < 2\varepsilon \tag{22}$$

From (21), (22) and Lemma 2 it follows that

$$|R^{-1}| \le M\,(1 - 2\varepsilon M)^{-1} \text{ for } 2\varepsilon M < 1. \tag{23}$$

We have

$$|R^{-1}-R_\varepsilon^{-1}| = |R_\varepsilon^{-1}\ (R_\varepsilon-R)\,R^{-1}| \le |R_\varepsilon^{-1}|\ |R-R_\varepsilon|\ |R^{-1}| \le 2\varepsilon M^2(1-2\varepsilon M)^{-1}. \tag{24}$$

Inequality (8) follows from (24). This completes the proof of Theorem 1.

§7. Some problems of approximation theory.

1. Suppose that $f(x)\varepsilon C_1$ is a given function, $C_1 = C_1(I)$,

$I = [-1, 1]$, $\|f\| = \max\limits_{x\varepsilon I} |f(x)|$, $E_n(f) = \min\limits_{P(x)\varepsilon P_n} \|f(x) - P(x)\|$,

P_n is the set of all polynomials of degree $\le n$, $w(f;t) =$

$= \sup\limits_{|x-y|\le t,x,y\varepsilon I} |f(x) - f(y)|$, $T_n(x) = 2^{-(n-1)} \cos(n \arccos x)$,

$x_j = x_{j,n+1} = \cos \frac{2j-1}{2h+2}\pi$, $1 \le j \le n+1$,

$$l_j(x) = l_{nj}(x) = \frac{T_{n+1}(x)}{T'_{n+1}(x_j)(x-x_j)},$$

$$L_n(x;f) = \sum_{j=1}^{h+1} f(x_j)\,l_j(x), \tag{1}$$

$$L_n^{(\delta)}(x,f_\delta) = \sum_{j=1}^{n+1} f_\delta(x_j)\ l_j(x) \tag{2}$$

$$\lambda_n = \Big\|\sum_{j=1}^{n+1} |l_j(x)|\ \Big\|, \tag{3}$$

$$\lambda'_n(x) = \sum_{j=1}^{h+1} |l'_j(x)|, \tag{4}$$

$$\lambda'_n = \|\lambda'_n(x)\|. \tag{5}$$

Given a set of numbers $f_\delta(x_j)$, $1 \le j \le n+1$, $|f_\delta(x_j) - f(x_j)| \le \delta$ we are going to find a stable approximation for $f(x)$ and give an error estimate of the formula for approximation.

Theorem 1. If $f \varepsilon C_1$ then

$$\|f(x) - L_n(x,f)\| \le (1+\lambda_n)E_n(f), \tag{6}$$

$$\|f'(x) - L_n'(x,f)\| \le (1+\lambda_n')E_{n-1}(f'), \tag{7}$$

$$|f'(x) - L_n'(x,f)| \le \left(1 + \frac{n\lambda_n}{\sqrt{1-x^2}}\right) E_{n-1}(f'), \tag{8}$$

$$\lambda_n' \le n^2\lambda_n;\ \lambda_n' \le \frac{n\lambda_n}{\sqrt{1-x^2}} \tag{9}$$

$$\|f^{(k)}(x) - L_n^{(k)}(x,f)\| \le C\, w(n^{-1}; f^{(r)})n^{k-r}(1+n^k \ln n),\ k \le r. \tag{10}$$

Theorem 1'. If $f \varepsilon C_r$, $r > 3$, then there exists $n(\delta)$ such that

$$\|L_{n(\delta)}^{(\delta)}(x,f_\delta) - f\| = 0(\delta \ln(\delta^{-1})),\ \delta \to 0, \tag{11}$$

$$\left\|\frac{dL_{n(\delta)}^{(\delta)}(x,f_\delta)}{dx} - f'\right\| = 0\left(\delta^{\frac{r-3}{r-2}} \ln(\delta^{-1})\right),\ \delta \to 0 \tag{12}$$

$$\left|\frac{d}{dx} L_{n(\delta)}^{(\delta)}(x,f_\delta) - f'\right| \le E_{n-1}(f')(1+\lambda_n'(x)) + \delta\lambda_n'(x), \tag{13}$$

$$\|L_{n(\delta)}^{(\delta)}(x,f_\delta) - f(x)\| \le (1+\lambda_n)E_n(f) + \delta\lambda_n, \tag{14}$$

Proofs of theorems 1 and 2 are given in [21]. The following Lemma is used in the proof and also is of independent interest.

Lemma 1. Suppose that $P_j(x) \varepsilon \mathcal{P}_n$ and

$$\sum_{j=1}^{m} |P_j(x)| \le M,\ x \varepsilon I, \tag{15}$$

where m is arbitrary. Then

$$\sum_{j=1}^{m} |P_j'(x)| \leq \min \left(\frac{Mn}{\sqrt{1-x^2}}, Mn^2\right). \qquad (16)$$

2. The following approximation problem is of interest in optics and electrodynamics ([19],[29]).

Let $D \subset R^r$ and $\Delta \subset R^r$ be some simply connected and connected domains, $f(x) \in C(\Delta)$ or $f(x) \in L^2(\Delta)$. Denote by W_D the class of functions which can be represented in the form

$$g(x) = (2\pi)^{-N/2} \int_D \exp\{-i(x,y)\} h(y)dy, \ h \in L^2(D) \qquad (17)$$

By $|\cdot|$, $\|\cdot\|$ we denote $C(\Delta)$ and $L^2(\Delta)$ norm respectively.

Problem A. How to find $f_\varepsilon \in W_D$ such that $|f-f_\varepsilon| < \varepsilon$?

Problem A. How to find $f_\varepsilon \in W_D$ such that $\|f-f_\varepsilon\| < \varepsilon$?

Let D = meas D,

$$g_n(y) = \left[\frac{1}{|D|} \int_D \exp\{-\frac{i}{2n+N} (t,y)\}dt\right]^{2n+N} \left(1 - \frac{|y|^2}{R^2}\right)^n \left(\frac{n}{\pi R^2}\right)^{N/2}. \qquad (18)$$

Here $|D|$ = meas D, $R > 0$ is such a number that the ball $|y| \leq R$ contains all vectors $t - y$, $t, y \in \Delta$, the origin in R_t^N is placed in the gravity center of D so that

$$\int_D t \, dt = 0. \qquad (19)$$

We assume that $\Delta = \bar{\Delta}$ and $D = \bar{D}$ are bounded simply connected and connected domains. Let

$$f_n(x) = \int_\Delta g_n(x-y) f(y) dy . \qquad (20)$$

Theorem 2. There exists $n = n(\varepsilon)$ such that $f_n(x)$ is the solution to problem A for $n \geq n(\varepsilon)$ if $f(x) \in C(\Delta)$ and to problem B if $f \in L^2(\Delta)$.

Theorem 3. If $|f(x)| \leq a$, $|\nabla f(x)| \leq b$, then

$$|f - f_n| \leq \frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2})} \frac{bR}{\sqrt{n}} + 0\ (\frac{1}{n}) \text{ as } n \to \infty,$$

$\Gamma(x)$ being Gamma function.

Proof of theorem 2. Note that $g_n(y) \varepsilon W_D$. Hence $f_n(x) \varepsilon W_D$. Let us prove that

$$[\frac{1}{|D|} \int_D \exp\{- \frac{i}{2n+N} (t,y)\} dt]^{2n+N} \to 1 \text{ as } n \to \infty, \tag{21}$$

$$(\frac{n}{\pi R^2})^{N/2} \int_\Delta (1 - \frac{|y-x|^2}{R^2})^n f(y) dy \to f(x) \text{ as } n \to \infty \tag{22}$$

uniformly in Δ if $f(x) \varepsilon C(\Delta)$.

We have

$$(2n+N) \ln \frac{1}{|D|} \int_D \exp\{- \frac{1}{2n+n} (t,y)\} dt = (2n+N) \cdot$$

$$\cdot \ln [1 - \frac{i}{2n+N} (y, \frac{1}{|D|} \int_D t dt) + \alpha_n], \quad \alpha_n = 0(n^{-2}) \tag{23}$$

From here and (19) we get (21). To prove (22) we can assume without loss of generality that $R = 1$, otherwise we can use scaling. For any η, $0 < \eta < 0.5$ we have

$$\lim_{n\to\infty} (\frac{n}{\pi})^{N/2} \int_{\eta \leq |x-y| \leq 1} [1 - |x-y|^2]^n dy = 0, \tag{24}$$

$$\lim (\frac{n}{\pi})^{N/2} \int_{|x-y|<\eta} [1 - |x-y|^2]^n dy = 1. \tag{25}$$

Further

$$f_n - f = J_1 + J_2 \equiv (\frac{n}{\pi})^{N/2} \int_\Delta [1 - |x-y|^2]^n [f(y) - f(x)] dy +$$

$$+ f(x) [(\frac{n}{\pi})^{N/2} \int_\Delta \{1 - |x-y|^2\}^n dy - 1], \tag{26}$$

$$|J_2| \leq |f| \cdot |(\frac{n}{\pi})^{N/2} \int_{|u| \leq 1} (1 - u^2)^n du - 1| \to 0 \text{ as } n \to \infty, \tag{27}$$

$$|J_1| < (\frac{n}{\pi})^{N/2} \int_{|u| \leq \varepsilon} (1-|u|^2)^n |f(u+x)-f(x)| du +$$

$$+ 2|f| (\frac{n}{\pi})^{N/2} \int_{\varepsilon < |u| \leq 1} (1-|u|^2)^n du \tag{28}$$

From (28) it follows that $J_1 \to 0$ as $n \to \infty$. We proved the first statement of theorem 2. To prove the second statement we must prove that

$$\|(\frac{n}{\pi R^2})^{N/2} \int_\Delta (1 - \frac{|x-y|^2}{R^2})^n f(y)dy - f(x)\| \to 0 \text{ as } n \to \infty. \tag{29}$$

As above we assume that $R = 1$. Let us fix $\varepsilon > 0$ and find $\phi(x) \varepsilon C(\Delta)$ such that $\|f-\phi\| < \varepsilon$. Setting

$$Q_n f \equiv (\frac{n}{\pi})^{N/2} \int_\Delta [1 - |x-y|^2]^n f(y)dy \tag{30}$$

we have

$$\|Q_n f - f\| \leq \|Q_n(f - \phi)\| + \|Q_n\phi - \phi\| + \|f - \phi\|. \tag{31}$$

Since $\|F\| \leq C|F|$, $C = C(\Delta)$ we obtain

$$\|Q_n\phi - \phi\| \to 0 \text{ as } n \to \infty. \tag{32}$$

It remains to prove that

$$\|Q_n\psi\| \to 0 \text{ as } n \to \infty, \ \psi \equiv f - \phi, \ \|\psi\| < \varepsilon. \tag{33}$$

We have

$$\|Q_n\psi\|^2 = \int_\Delta dx\{(\frac{n}{\pi})^{N/2} \int_\Delta [1 - |x-y|^2]^n |\psi(y) dy (\frac{n}{\pi})^{N/2} \cdot$$

$$\cdot \int_\Delta (1 - |x-z|^2]^n |\psi(z)| dz\} \leq \frac{1}{2}(\frac{n}{\pi})^N \int_{|u| \leq 1, |v| \leq 1} dudv(1 - |u|^2)^n$$

$$\{\int_\Delta (|\psi(x+u)|^2 + |\psi(x+v)|^2)dx\} \leq$$

$$\leq \varepsilon (\frac{n}{\pi})^N \int_{|u| \leq 1 |v| \leq 1} (1 - |u|^2)^n (1 - |v|^2)^n dudv \leq \text{const } \varepsilon. \tag{34}$$

where const does not depend on n.

Proof of theorem 3. To prove this theorem we must sharpen some of the estimates given in proof of theorem 1. First we note that α_n in (23) can be estimated:

$$|\alpha_n| \le \frac{c}{2|D|(2n+N)^2} \int_D |(t,y)|^2 dt \le \frac{c_1}{(2n+N)^2} \tag{35}$$

From (19) and (7) it follows that

$$|[\frac{1}{|D|} \int_D \exp\{- \frac{i}{2n+N} (t,y)\}dt]^{2n+N} - 1| = 0 (\frac{1}{n}) \text{ as } n \to \infty \tag{36}$$

Further

$$(\frac{n}{\pi R^2})^{N/2} \int_\Delta (1 - \frac{|y|^2}{R^2})^n dy = 1 + 0 (\frac{1}{n}) \text{ as } n \to \infty \tag{37}$$

Now we estimate integral J_1 from formula (10) without assumption $R = 1$. We have

$$J_1 \le b (\frac{n}{\pi R^2})^{N/2} \int_{|u| \le R} (1 - \frac{|u|^2}{R^2})^n |u| du =$$

$$= bR (\frac{n}{\pi})^{N/2} S_N \int_0^1 (1 - v^2)^n v^N dv =$$

$$= \frac{bR\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2}) \sqrt{n}} + 0 (\frac{1}{n^{3/2}}) \text{ as } n \to \infty, \tag{38}$$

where $S_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of the unit sphere in R^N, itegral $\int_0^1 (1 - v^2)^n v^N dv = \frac{1}{2}B (\frac{N+1}{2}, n + 1)$, $B(x,y)$ is beta-function. We can evaluate this integral as $n \to \infty$ either using Stirling formula or Laplace method for asymptotic evaluation of integrals. As a result we get (20).

3. Approximation of derivatives.

Let $f(x) \varepsilon C_2$, $f_\delta \varepsilon C_0$, $C_n = C_n[0,1]$ is the Banach space with the norm

$$\|f\|_n = \sum_{j=0}^{n} \max_{0\le x\le 1} |f^{(j)}(x)|, \quad \|\cdot\| = \|\cdot\|_0, \quad \|f - f_\delta\| \le \delta.$$

Given f_δ how to construct $u_\delta(x)$ so that $\varepsilon(\delta) \equiv \|u_\delta - f'\| \to 0$ as $\delta \to 0$? This is the problem of stable differentiation. Let

$$\Delta_h f = \begin{cases} h^{-1}[f(x+h) - f(x)], \ 0 \le x \le \frac{h}{2} & , \\ h^{-1}[f(x+\frac{h}{2}) - f(x-\frac{h}{2}), \ \frac{h}{2} \le x \le 1 - \frac{h}{2} & , \\ h^{-1}[f(x) - f(x-h)], \ 1 - \frac{h}{2} \le x \le 1. \end{cases}$$

Theorem 4. Suppose that $\|f''\| \le M$, $h(\delta) = 2(\delta M^{-1})^{1/2}$. Then

$$\|f' - \Delta_{h(\delta)} f\| \le 2(M\delta)^{1/2}. \qquad (39)$$

Proof of Theorem 4. We have $\|\Delta_h f_\delta - f'\| \le \|\Delta_h(f_\delta - f)\| + \|\Delta_h f - f'\| \le 2\delta h^{-1} + 0.5Mh$. Minimizing in $h > 0$ we find $h = h(\delta) = 2(\delta M^{-1})^{1/2}$ and the estimate $\|f' - \Delta_{h(\delta)} f_\delta\| \le 2(M\delta)^{1/2}$.

Remark 1. Suppose that $f \varepsilon C_{2n-1}$, $\Delta_h^n f = h^{-1} \sum_{k=-n}^{n} A_k^{(n)} f(x+khn^{-1})$.

If coefficients $A_k^{(n)}$ are determined by the system

$$\sum_{k=-n}^{n} k^p A_k^{(n)} = n\delta_{p1}, \ 0 \le p \le 2n-1, \ \delta_{p1} = \begin{cases} 0, p\ne 1 \\ 1, p=1 \end{cases} \qquad (40)$$

then $|\Delta_n^h f - f'| \le M_n h^{2n1}$. We can estimate $\|\Delta_h^{(n)} f_\delta - f'\|$ and find the optimal $h = h(\delta)$.

Suppose now that $u = f + n$, where $u = u(t)$ is a random function, f is a signal and n is noise. The problem is to find a linear operator (estimate) L such that $D[Lu - f'] = \min$, D is the

symbol of dispersion. The solution of this problem was obtained by N. Wiener for stationary random processes [23]. This solution is difficult to find analytically and to apply in practice. We give here a quasioptimal solution of this problem. The idea is to look for a solution in a subset of the set of all linear estimates. We give an estimate which is easy to calculate, easy to apply in practice and which has an error of the same order as optimal estimate of f'. Let us look for an estimate of the form

$$Lu = \Delta_h^{(n)} u = h^{-1} \sum_{k=-m}^{m} A_k^{(m)} u(t + khm^{-1}) \qquad (41)$$

where coefficients $A_k^{(n)}$ are determined by system (40),

$A_{\pm 1}^{(1)} = \mp 0.5$, $A_0^{(1)} = 0$; $A_{\mp}^{(2)} = \pm 1/6$,

$A_{\mp 1}^{(2)} = \mp 4/3$, $A_0^{(2)} = 0$; $A_{\mp 3}^{(3)} = \mp 0.3$,

$A_{\mp 2}^{(3)} = \pm 0.9$, $A_{\mp 1}^{3} = \mp 2.25$, $A_0^{(3)} = 0$; $A_{\mp 4}^{(4)} = \pm 1/70$,

$A_{\mp 3}^{(4)} = \mp 16/105$, $A_{\mp 2}^{(4)} = \pm 4/5$, $A_{\mp 1}^{(4)} = \mp 16/5$, $A_0^{(4)} = 0$.

Parameter h is to be found from the condition $D[\Delta_h^{(m)} u - f'] = \min$.

Suppose that f and n are uncorrelated, R_f, R_n are covariance functions of f and n respectively. Suppose that f and n are stationary random processes and $n = \delta v$, $\delta > 0$, $D[v] = 1$, $\delta^2 = D[n]$, $R_n = \delta^2 R_v$.

We have

$$D[\Delta_h^{(n)} u - f'] = D[\Delta_h^{(m)} f - f'] + D[\Delta_h^{(m)} n],$$

$$D[\Delta_h^{(m)} h] = h^{-2}\delta^2 \sum_{j,p=-m}^{m} A_j^{(m)} A_p^{(m)} R_v(h(j-p)m^{-1}) \equiv \lambda_m h^{-2}\delta^2$$

$$D[\Delta_h^{(m)} f - f'] \leq \mu_m h^{m-2},$$

where

$$\mu_m = D[f^{2m-1}] \left(\sum_{k=-m}^{m} \frac{|km^{-1}|^{2m-1} |A_k^{(m)}|}{(2m-1)!} \right)^2 .$$

Therefore the optimal $h = h(\delta)$ is to be found from the condition

$$\varepsilon \equiv \lambda_m h^{-2}\delta^2 + \mu_m h^{4m-2} = \min.$$

Thus $h_{opt} = [\lambda_m \mu_m^{-1} (2m-1)^{-1}]^{1/(4m)} \delta^{1/(2m)}, \varepsilon = 0 \ (\delta^{2-\frac{1}{m}})$.

These results were used for finding extremal points of random functions. If a random signal $u = f + \delta v$, where v is noise, and $f \varepsilon C^2$, $|f''| \leq M$, is a deterministic function defined on a segment $[a,b]$ and unimodal (it means that f has only one extremum t^* in the segment $[a,b]$), then the following algorithm for finding t_0 is convenient in practice. Let us take $h = 2(\delta M^{-1})^{1/2}$ and $y_j = (2h)^{-1} [u(t_j + h) - u(t_j - h)]$, $j = 1, 2, \ldots, t_j = a + jh$. If $y_j \ y_{j+1} < 0$ then we assume that $t_j < t^* < t_{j+1}$.

4. Optimal harmonic synthesis.

Suppose that the system $\{\phi_n(x)\}$ forms an orthomormal basis of $L^2(D)$, $f(x) = \sum_1^\infty a_n \phi_n(x)$, and instead of a_n the numbers $c_n = a_n + \delta_n$ are known, where $\overline{\delta_n \delta_m^*} = \sigma_n^2 \delta_{nm}$. Here the bar means statistical averaging, δ_{nm} is the Kronecker symbol, star means complex conjugation. We assume that $\sigma_n^2 \leq \sigma^2$, $n = 1,2,\ldots$. The problem is to find $f(x)$ with minimal error if $\{c_n\}$ are given.

We give two approaches to the problem. The first one consists in finding $N = N(\sigma)$ from the condition

$$\varepsilon_n(\sigma) \equiv \overline{\|f(x) - \sum_1^n C_n\phi_n(x)\|^2} = \min, \quad \|\cdot\| = \|\cdot\|_{L^2(D)} \tag{42}$$

The second approach consists in introducing some convergence multipliers $\rho_n(q)$ and finding $q = q(\sigma)$ from the condition

$$\overline{\|\sum_1^\infty [\rho_n(q)\ c_n - a_n]\phi_n(x)\|^2} = \min. \tag{43}$$

We assume that

$$|a_n| \le An^{-a}, \ a > 0.5. \tag{44}$$

The answer to problem (42) is the following:

$$N(\sigma) = (A\sigma^{-1})^{1/a}, \varepsilon_{\min}(\sigma) \le \sigma^{2-a^{-1}} A^{a^{-1}} \frac{2a}{2a-1}. \tag{45}$$

If we take $\rho_n(q) = \exp(-qn)$ then the solution to problem (43) gives q_{opt} which is the positive solution of equation

$$A^2\ \zeta(1+\varepsilon)cq^{c-1} = \frac{2\sigma^2\exp(-2q)}{\{1-\exp(-2q)\}^2}, \ \zeta(x) \equiv \sum_{n=1}^\infty n^{-x}, \tag{46}$$

$c = 2$ if $a > \frac{3+\varepsilon}{2}$, $c = 2a-1-\varepsilon$ if $2a<3+\varepsilon$. Here $0 < \varepsilon < 2a-1$.

Generalizations, examples, applications to signal processing and numerical results are given in [33].

§8. Optimal solution of antenna synthesis problems.

1. Consider the problem of finding the current distribution in a linear antenna from the antenna pattern. This problem can be reduced to integral equation

$$Aj = \int_{-\ell}^{\ell} j(z)\ \exp(ikz)dz = f(k), \ -k_0 \le k \le k_0, \tag{1}$$

where $f(k)$ is the given radiation pattern, $j(z) \varepsilon L^2 = L^2(-\ell,\ell), j(z)$ is the desired current distribution, $j(z)$, $f(k)$ are scalar functions. It is quite clear that equation (1) has no more than one solution and this equation has the solution iff $f(k) \varepsilon W_\ell = W_{(-\ell,\ell)}$ where class W_D was defined in §7, section 2. If the given function $f(k) \notin W_\ell$ but $f_\varepsilon(k) \varepsilon W_\ell$, $\|f_\varepsilon - f\| < \varepsilon, \|\cdot\| = \|\cdot\|_{L^2(-k_0,k_0)}$, then we consider j_ε corresponding to f_ε as an approximate solution to synthesis problem. As operator A is compact small perturbations of f can imply large variations of j. So j_ε can change much when f_ε is changed a little. This phenomenon was discussed in the literature (Minkovich-Jakovlev [1]) in connection with the superdirectivity of antennas. From practical point of view we should find a stable current distribution which generates radiation pattern close to the desired pattern $f(k)$.

So we require that

$$\int_{-\ell}^{\ell} |j|^2 dz \le M_0, \int_{-\ell}^{\ell} |j'(z)|^2 dz \le M_1, \tag{2}$$

where M_0, M_1 are some constants.

The set of functions satisfying conditions (2) we denote by $\Omega(M_0, M_1)$. It is convex and compact in L^2. So we came to the problem of solving equation (1) under conditions (2) in the following sense: given a function $g(k) \varepsilon L^2(-k_0,k_0) = L^2$ we look for $f(k) \varepsilon W_\ell$ such that

$$J = \int_{-\infty}^{\infty} |g(k) - f(k)|^2 dk = \min, \; g(k) = 0 \text{ for } |k| > k_0, \tag{3}$$

$$f(k) = Aj \text{ (see (1))}, \tag{4}$$

$$j \varepsilon \Omega(M_0, M_1). \tag{5}$$

This optimization problem can be solved in principle by non-linear programming methods, by direct methods of calculus of variations and by method of calculus of variations based on the Euler equation. We consider these possibilities.

2. Let

$$\tilde{g}(z) = (2\pi)^{-1} \int_{-\infty}^{\infty} g(k)\ \exp(-ikz)dk \qquad (6)$$

By Parseval equality we get:

$$\frac{J}{2\pi} = \int_{-\infty}^{\infty} |g(z) - j(z)^2 dz = \int_{|z|>\ell} |\tilde{g}(z)|^2 dt + \int_{-\ell}^{\ell} |\tilde{g}(z) - j(z)|^2\ dz =$$

$$= \delta(g,\ell) + J_1, \qquad (7)$$

where $\delta(g,\ell)$ does not depend on $j(z)$. Hence J and J_1 reach their minimum at the same function and problem (3)-(5) is equivalent to the problem:

$$\begin{cases} J_1 = \min \\ j\varepsilon\Omega(M_0,\ M_1) \end{cases} \qquad (8)$$

We assume at first for simplicity that

$$j(\ell) = j(-\ell). \qquad (9)$$

In the end we eliminate this assumption. Let

$$j(z) = \sum_{n=-\infty}^{\infty} j_n \frac{\exp(in\pi z/\ell)}{\sqrt{2\ell}},\ \tilde{g} = \sum_{n=-\infty}^{\infty} \tilde{g}_n \frac{\exp(in\pi z/\ell)}{\sqrt{2\ell}}, \qquad (10)$$

$$j_n \equiv \int_{-\ell}^{\ell} j(z) \frac{\exp(-in\pi z(\ell)}{\sqrt{2\ell}} dz.$$

We note that assumption (9) will be used only once: it allows to differentiate the Fourier series of $j(z)$ termwise. Substituting (10) in (8) we get

$$\begin{cases} J_1 = \sum\limits_{h=-\infty}^{\infty} |j_n - \tilde{g}_n|^2 = \min \\ \sum\limits_{n=-\infty}^{\infty} |j_n|^2 \le M_0, \quad \sum\limits_{n=-\infty}^{\infty} n^2 |j_n|^2 \le M_1 \dfrac{\ell^2}{\pi^2}. \end{cases} \tag{11}$$

It is the problem of convex programming. Since Ω (M_0, M_1) is uniformly convex and compact in ℓ^2 and functional $J_1 = \|\tilde{g} - j\|^2$ is strictly convex problem (11) has a solution and the solution is unique. A numerical solution to problem (11) can be obtained so. We fix N and set:

$$\begin{cases} J_{1N} = \sum\limits_{n=-N}^{N} |j_n - \tilde{g}_n|^2 \\ \sum\limits_{n=-N}^{N} |j_n|^2 \le M_0, \quad \sum\limits_{n=-N}^{N} n^2 |j_n|^2 \le M_1 \dfrac{\ell^2}{\pi^2} \end{cases} \tag{12}$$

This is a finite dimensional problem of convex programming. Denote by $j^{(N)}$ the solution to problem (12) and show that $j^N \to j$ as $N \to \infty$. Here j is the solution to problem (8) and $\to$ denotes convergence in ℓ^2. We denote $J_{1N}(j^N) = \delta_N$,

$$J_1(j) = d, \quad \sum_{n=-N}^{N} |j_n - \tilde{g}_n|^2 = d_N, \quad \sum_{|n|>N} |\tilde{g}_n|^2 = \varepsilon_N.$$

Since $d \le d_N + \varepsilon_N$, $\delta_N \le d_N$ we get $\lim\limits_{N\to\infty} \delta_N \le d$. It is clear that

$\delta_N \le \delta_{N+1}$. Let $\delta = \lim\limits_{N\to\infty} \delta_N$, $\delta \le d$, $\lim\limits_{N\to\infty} j^N = j^{(\infty)} \in \Omega(M_0, M_1)$,

$J_1(j^{(\infty)}) = \delta \le d$. Hence by definition of d we get $\delta = d$ and by uniqueness of solution to problem (8) we get $j^{(\infty)} = j$. Hence $j^{(N)} \to j$ as $N \to \infty$. If $\tilde{g} \in \Omega(M_0, M_1)$ then the solution to problem (11) is trivial: $j_n = \tilde{g}_n$ but it is not interesting since in practice $\tilde{g} \notin \Omega(M_0, M_1)$. If $\tilde{g} \notin \Omega(M_0, M_1)$ then from geometrical reasons

it is clear that the solution to problem (11) lies on the boundary of the set $\Omega(M_0,M_1)$ and hence at least in one of inequalities (11) should be equality. If one of inequalities (11) is strict and the other is equality we can omit the strict inequality and consider the problem (11) with only one restriction in the form of equality. This relatively simple problem we do not want to discuss and pass over to the more difficult case when both inequalities (11) are equalities. So we get the following problem

$$\begin{cases} J_1 = \sum_{n=-\infty}^{\infty} |j_n - \tilde{g}_n|^2 = \min \\ \sum_{n=-\infty}^{\infty} |j_n|^2 = M_0, \quad \sum_{n=-\infty}^{\infty} n^2|j_n|^2 = M_2 = M_1 \frac{\ell^2}{\pi^2} \end{cases} \tag{13}$$

The same is valid for problem (12), for which we have

$$\begin{cases} j_{1N} = \sum_{h=-N}^{N} |j_n - \tilde{g}_n|^2 = \min \\ \sum_{n=-N}^{N} |j_n|^2 = M_0, \quad \sum_{n=-N}^{N} n^2|j_n|^2 = M_2. \end{cases}$$

The latter problem can be easily solved with the help of the Lagrange multipliers. We set

$$\Phi = J_{1N} + \lambda \left(\sum_{n=-N}^{N} |j_n|^2 - M_0\right) + \mu\left(\sum_{n=-N}^{N} n^2|j_n|^2 - M_2\right) \tag{15}$$

and write the Euler equations

$$\frac{\partial \Phi}{\partial j_n} = 0, \quad \frac{\partial \Phi}{\partial j_n^*} = 0 \quad - N \le n \le N \tag{16}$$

System (16) can be easily solved:

$$j_n = \frac{\tilde{g}_n}{1+\lambda+\mu n^2}, \quad - N \le n \le N. \tag{17}$$

Parameters λ, μ should be found from two equations

$$\sum_{n=-N}^{N} \frac{|\tilde{g}_n|^2}{(1+\lambda+\mu n^2)^2} = M_0, \quad \sum_{n=-N}^{N} \frac{n^2 |g_n|^2}{(1+\lambda+\mu n^2)^2} = M_2 \tag{18}$$

Equation (18) can be easily solved by numerical methods.

3. Problem (13) can be written so

$$\begin{cases} J_1 = \int_{-\ell}^{\ell} |j(z) - \tilde{g}(z)|^2 dz = \min \\ \int_{-\ell}^{\ell} |j|^2 dz = M_0, \ \int_{-\ell}^{\ell} |j'(z)|^2 dz = M_1 \end{cases} \tag{19}$$

Using calculus of variations we get the Euler equation and the natural boundary conditions:

$$(1+\lambda)j(z) - \mu j''(z) = \tilde{g}(z), \ -\ell \le z \le \ell, \ j'(-\ell) = j'(\ell) = 0 \tag{20}$$

This equation can be solved explicitly and then parameters λ, μ can be found in principle from equalities (19). Problem (19) can be also solved by direct methods of calculus of variations, for example by Ritz methods. If we set

$$j^{(N)} = \sum_{n=-N}^{N} j_n \frac{\exp(in\pi z/\ell)}{\sqrt{2\ell}},$$

we get system (16) for unknown coefficients j_n with restrictions (14).

4. Here we eliminate assumption (9). If $j(\ell) \neq j(-\ell)$ then even for a smooth in the closed segment $[-\ell,\ell]$ function its Fourier coefficients are $O(\frac{1}{n})$ and so the Fourier coefficients of $j_n'(z)$ are not equal to $\frac{i\pi n}{\ell} j_n$. It is so because the 2ℓ-periodic continuation of the function $j(z)$ is not smooth. To avoid this difficulty we can use the following orthonormal basis in $L^2(-\ell,\ell)$:

$$\phi_n = \frac{1}{\sqrt{\ell}} \cos\left(\frac{n\pi z}{2\ell} + \frac{\pi n}{2}\right), \ n = 0, 1, 2... \tag{21}$$

Note that ϕ_n are the eigenfunctions of the following problem

$$y'' + \lambda y = 0, \; -\ell \le z \le \ell; \; y'(-\ell) = y'(\ell) = 0. \tag{22}$$

Fourier coefficients

$$C_n = \int_{-} j(z)\phi_n(z)dz \tag{23}$$

are $0\left(\frac{1}{n^2}\right)$ as $n \to \infty$ and we can differentiate Fourier series of $j(z)$ according to the system $\{\phi_n\}$ termwise.

6. By the given in this section method one can solve other antenna synthesis problem, for example, the problem of synthesis of the spherical antenna and directorial antenna.

§9. Stable solution of integral equation of the first kind with logarithmic kernel.

1. Consider the equation

$$Af = \int \ln(r_{xy}^{-1}) \, f(y)dy = g(x), \; x\in D, \; \int \equiv \int_D, \tag{1}$$

where $D\subset R^m$ is a bounded surface. Because $\ln r_{xy}^{-1}$ changes sign, it is possible that for some D the homogeneous equation (1) (equation (1_0)) will have a nontrivial solution. Let us assume that (A): Equation (1) is solvable in $H = L^2(D)$ and equation (1_0) has a nontrivial solution in H. We give an iterative process for calculation of solutions of equations (1),(1_0). The method given holds for any A semibounded below and such that its kernel $A(x,y) > -k$, $k = \text{const}$.

2. Let us take a number $d > \text{diam } D$. Then $\ln(dr_{xy}^{-1}) > 0$ for $x,y\in D$. Equation (1) is equivalent to equation

$$Bf \equiv (2\pi)^{-1} \int \ln(dr_{xy}^{-1}) \, f(y)dy = h(x) \equiv g(x) + \ln d \int f(y)dy \equiv g+c(f) \tag{2}$$

Operator B is positive in H. Thus $Bf = 0$ implies that $f = 0$. (3)

Lemma 1. $\dim \text{Ker } A = 1$.

Proof of Lemma 1. Suppose $f_0, f_1 \in \text{Ker } A$.

Then $\int f_j dx \neq 0$, $j = 0, 1$. Indeed if $\int f dx = 0$, $Af = 0$, then $Bf = 0$ and thus $f = 0$. Let us take $b > 0$ such that $\int (f_0 - bf_1) dx = 0$. Then $f_0 = bf_1$. This completes the proof.

According to Lemma 1 the problem $Af_0 = 0$, $\int f_0 dx = 1$ has a unique solution. If equation (1) is solvable in M, it has a solution which satisfies the condition

$$\int f dx = 0, \tag{4}$$

and thus $Bf = g$. Indeed, if $Af_1 = g$, then $f = f_1 - af_0$, where $a = \int f_1 dx$, $Af_1 = Af = Bf = g$.

3. Let us look for the solution of (1) which satisfies (4). This problem has the unique solution for $g \in R(A)$ and is equivalent to the problem

$$Bf = g. \tag{5}$$

The kernel of equation (5) is pointwise positive. Therefore the theory given in §3 is applicable and we have Theorem 1. Let

$$a(x) = \{\int \ln (dr_{xy}^{-1}) dy\}^{-1}, \quad f = a\psi,$$

$$\psi_{n+1} = (I - B_1)\psi_n + g, \quad \psi_0 = g, \tag{6}$$

where $B_1\psi \equiv B(a\psi)$. Then $||\psi - \psi_n|| \to 0$ as $n \to \infty$.

Remark 1. Iterative process (6) allows us to aclculate the solution of (5). Because $B > 0$ in H, there is a possibility to use iterative process

$$\psi_{n+1} = (I - kB)\psi_n + kg, \quad 0 < k < 2||B||^{-1} \tag{7}$$

(see (4.12)).

10. Bibliographical note

1. Much more examples of ill-posed problems can be given.
2. The result given in Theorem 1 is similar to some well-known results.
3. Theorem 1 was announced in [24].
4. Theorem 1 was proved in [25].
5. The results of this section appeared in slightly different form in [26] and were used in [27]-[28].
6. The theory outlined here was developed in [15]-[19]. The results of this section are new.
7. The results of this section were given in [29]-[33].
8. The results of this section were presented in [34], [35]. Similar theory is of interest in optics [29].
9. The result of this section is new.

The nonlinear antenna synthesis problem was sutided in [30]. We do not discuss in this paper some known conceptions in treating of ill-posed problems: Regularization method [1], [3], [5], method of quasiinversion [2], method of quansisolution [4]. The iterative process for solution of integral equation of the first kind with selfadjoint nonnegative operator first was given in [44] (see also [6]-[9]). The inverse scattering problem was treated in [11]. In [14] the inverse problem of potential theory was studied. In [37] the stability of calculations in variational methods is studied. In [38], [39] inverse diffraction problem was studied. In [40] a method of stable solution of some ill-posed problems is given and applications to partial differential equations are considered. In [41] many inverse problems, which

are incorrectly posed, are discussed. Among them there are problems of interest in geophysics, optics and quantum, mechanics. In [42] statistical approach to solution of ill-posed problems is discussed. Any linear solvable equation Af = g with a bounded operator A on a Hilbert space can be solved by an iterative process:

$$f_{n+1} = (I - \alpha A^*A)f_n + \alpha g, \ f_0 \varepsilon H, \ 0 < \alpha < 2 \|A^*A\|^{-1}.$$

One of the first results in this direction was given in [43]. We have no possibility to discuss here the statistical approach to the ill-posed problems [42], [43].

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9